Constraint Handling and Stability Properties of Model-Predictive Control

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Effects of hard constraints in the stability of model-predictive control (MPC) are reviewed. Assuming a fixed active set, the optimal solution can be expressed in a general state-feedback closed form, which corresponds to a piecewise linear controller for the linear model case. Changes in the original unconstrained solution by the active constraints and other effects related to the loss of degrees of freedom are depicted in this analysis. In addition to modifications in the unconstrained feedback gain, we show that the presence of active output constraints can introduce extra feedback terms in the predictive controller. This can lead to instability of the constrained closed-loop system with certain active sets, independent of the choice of tuning parameters. To cope with these problems and extend the constraint handling capabilities of MPC, we introduce the consideration of soft constraints. We compare the use of the l_2-(quadratic), l_1-(exact), and l_∞-norm penalty formulations. The analysis reveals a strong similarity between the control laws, which allows a direct extrapolation of the unconstrained tuning guidelines to the constrained case. In particular, the exact penalty treatment has identical stability characteristics to the correspondent unconstrained case and therefore seems well suited for general soft constraint handling, even with nonlinear models. These extensions are included in the previously developed Newton control framework, allowing the use of the approach within a consistent framework for both linear and nonlinear process models, increasing the scope of applications of the method. Process examples illustrate the capabilities of the proposed approaches.

Introduction

An important aspect in the application of model predictive control (MPC) is the effect of the presence of constraints in the stability of the resultant closed-loop system. Recent work by Zafiriou (1990–1991) has shown this to be a significant issue, because of the frequency with which saturation can occur during routine operation. Considering the QDMC (quadratic dynamic matrix control) framework, he was able to identify several situations where the decrease in the degrees of freedom available to influence the process masked the effect of the tuning parameters for stabilization, rendering the constrained system unstable. These effects lead therefore to an important issue in the design of constrained control systems.

Usually, current design of constrained predictive controllers is an iterative process. Typically, it starts with a candidate set of tuning parameters, representing an acceptable unconstrained design, obtained by taking into consideration, for example, the performance and robustness of the unconstrained system. The closed-loop performance of this controller then needs to be evaluated in the presence of the constraints which can potentially become active during the operation of the process. Depending on the results obtained, it might be necessary to adjust the initial parameters in order to achieve a good overall behavior with all possible active sets. These changes require in turn a reevaluation of the modified design according to the criteria used in the first step. Hence, it is essential in a good constrained design methodology to have the components to assess the closed-loop performance of the process, especially its stability properties in the presence of various types of constraints, and to have a systematic procedure for adjusting the tuning parameters, or if necessary to adapt the constraint han-

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dling methodology used in order to provide a global, closed-loop performance. Both of these issues are addressed in detail throughout this article.

We start by presenting a comprehensive treatment of the effects of the presence of active constraints in the stability of the predictive control. This study is motivated by the pioneering work of Zafiriou (1990; 1991a,b) in this area. Similar to this analysis, the nonlinearities introduced by the presence of active constraints are handled through separate consideration of each different active set. A disadvantage of this approach is that a complete analysis requires checking all possible combinations of constraints in order to guarantee global properties. This is essentially a combinatorial task, which introduces considerable difficulties in the analysis of predictive horizons of the size usually considered in practice. However, as described below, much of the same information can be derived from a significantly smaller subset of constraints. The present analysis also shows that the optimal solution can be expressed in a general state-feedback closed form, similar to the unconstrained case. Here the modifications introduced in the original solution by the presence of constraints are clearly displayed. Even so, a change in the adjustable parameters of the controller might not be sufficient to prevent, in the worst case, the instability of the closed-loop system with certain active sets, as illustrated in the examples included.

To deal with the stability problems caused by the presence of active hard constraints, we introduce in this work a constraint relaxation scheme based on the use of penalty functions. This feature is also created as an extension of the previously developed Newton control framework (Li and Biegler, 1989; Oliveira and Biegler, 1994), allowing a straightforward generalization of these results to the nonlinear case, because of the uniform treatment of both types of systems provided by the formalism.

This article is organized as follows: a brief review of the Newton control formulation is given and followed by the stability analysis of MPC in the presence of hard constraints. Assuming a fixed active set, we first show that the optimal input profile can be expressed in a general state-feedback form. In addition to changes in the unconstrained feedback structure, we demonstrate that the presence of active output constraints can introduce extra feedback terms in the predictive controller. Therefore, together with a decrease in the degrees of freedom available, this can lead to stability problems of the constrained system with certain active sets, independent of the choice of tuning parameters used. A constraint relaxation treatment capable of preventing the occurrence of these problems is introduced next. Here we address first the common case of a quadratic penalty objective for which simpler stability results are derived compared to the treatment of Zafiriou (1991a). These directly relate the stability of the relaxed constrained problem to an equivalent modification in the tuning parameters of the original unconstrained problem. We show that in general only a maximum finite value of the penalty parameter can be tolerated for stability. Since the stability characteristics of the relaxed problem are still dependent on the current active set, this approach suffers from the same disadvantage as the hard constraint analysis; it is combinatorial in the length of the predictive horizon.

To eliminate this last requirement we then introduce an alternative soft constraint treatment, that uses the l_1 or l_∞-norm penalty formulations. In this case, we demonstrate that the resultant constrained system has identical bounded stability properties to the corresponding unconstrained situation. Moreover, if the original hard constrained controller is stable, the l_1 strategy requires only a finite penalty parameter (larger than the norm of the Kuhn-Tucker multipliers) to match the solution of the original problem, in contrast with the quadratic penalty case which requires a parameter value of infinity. This characteristic allows much better control of the errors resulting from constraint relaxation, and simplifies considerably the use of the soft constraints, especially for nonlinear systems. We also discuss in this section the possibility of occurrence of steady-state offsets for large values of the penalty parameter and short horizons (due to the receding nature of the control law), together with sufficient conditions for their elimination. The use of alternative penalty formulations and their possible effects on the closed-loop stability properties are also considered here. Finally, these developments are illustrated with application to several process examples.

Preliminary Definitions

A short overview is presented here of the Newton control formulation used in the analysis of the stability properties of MPC in the next section; a more complete description can be found in Oliveira and Biegler (1994). The analysis presented here is based on the control law expressed in magnitude form, where the independent variable is U. This formulation is particularly convenient for studying the stability properties of the resultant controller.

We denote by $u \in \boldsymbol{R}^{n_i}$ the vector of system inputs (manipulations), $x \in \boldsymbol{R}^{n_s}$ the vector of state variables, $y \in \boldsymbol{R}^{n_o}$ the vector of system outputs, $\theta \in \boldsymbol{R}^{n_p}$ a vector of system parameters, and $d \in \boldsymbol{R}^{n_d}$ the vector of process disturbances. The lengths of the input and output predictive horizons are m and p, respectively, with $m \leq p$. The identity matrix of order n is denoted here by $I_n \in \boldsymbol{R}^{n \times n}$. We start by defining a discrete system operator as:

$$\chi_k = x_{k+1} = \chi(t_k + T;\ t_k,\ x_k,\ u_k,\ d;\ \theta),$$

where we assume that the transition function χ is continuous and differentiable with respect to all of its arguments. This operator can be obtained from a continuous plant model, provided that it satisfies the proper Lipschitz continuity conditions (Economou, 1985), or by direct application of discrete identification techniques to the system under consideration. In a similar form, we denote the operator induced by the control algorithm generically by $u_{k+1} = \psi(x_k, u_k, y_{rk})$, where y_{rk} corresponds to the value of an external reference input (setpoint), specified over a finite horizon (also in discrete time). Thus, together with the previous plant model, an augmented closed-loop system can be defined as:

$$z_{k+1} = \begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix} = X(T;\ z_k,\ y_{rk}) = \begin{bmatrix} \chi(x_k,\ u_k) \\ \psi(x_k,\ u_k,\ y_{rk}) \end{bmatrix}, \qquad (1)$$

where we have omitted the dependence of z on T, d, and θ for clarity of notation.

More specifically, the selection of the control law in the Newton framework is based on a quadratic performance index in a moving horizon of length p, which corresponds to the solution of the following constrained quadratic programming (QP) problem:

$$\min_U J_2 = (Y_{sp} - Y)^T Q_1 (Y_{sp} - Y) + (U - U_r)^T Q_2 (U - U_r) \quad (1a)$$

$$\text{s.t. } Y = Y^* + S_m U \quad (1b)$$

$$U_l \leq U \leq U_u \quad (1c)$$

$$Y_l \leq Y \leq Y_u. \quad (1d)$$

Note that this QP can also be expressed in terms of input changes ΔU, as in the QDMC formulation (Garcia and Morari, 1982). Here we simply substitute $L\Delta U$ for U above where L is a lower triangular matrix with elements of one on and below the diagonal, and the above QP as well as the entire analysis in the article remains unchanged. The solution of Eqs. 1a-1d corresponds to a Newton step towards the solution of the predictive control problem for a nonlinear process model, or the optimal profile in the linear case. Here $Q_1 = \text{diag}\{Q_{yi}\} \in \mathbf{R}^{(n_o p) \times (n_o p)}$ and $Q_2 = \text{diag}\{Q_{ui}\} \in \mathbf{R}^{(n_i m) \times (n_i m)}$ are adjustable weights in the objective. Also, capital letters E, X, Y, and U are used throughout this article to denote augmented vectors defined for the entire predictive horizon. Thus, for example, U corresponds to the augmented input vector:

$$U \in \mathbf{R}^{n_i m} \equiv [u_k^T \quad u_{k+1}^T \quad \cdots \quad u_{k+m-1}^T]^T.$$

The vector U_r defines a reference trajectory for the inputs, similar to the role of Y_{sp} for the outputs. S_m is the system *dynamic matrix*. This matrix can be formed directly from a linear process model or obtained from a sensitivity analysis of a nonlinear model around a nominal trajectory $\overline{U}$ (Oliveira, 1994). It generates in the second case a linear time-varying (LTV) approximation of the original process model, as part of the Newton iteration. Y^* corresponds to the system response for a zero input. For linear models, Y^* can be expressed directly in terms of the initial conditions and sensitivity coefficients, as:

$$Y^* = C^* x_k = \begin{bmatrix} C_{k+1}\Phi_k \\ C_{k+2}\Phi_{k+1}\Phi_k \\ \cdots \\ C_{k+p}(\Pi_{j=1}^{p}\Phi_{k+p-j}) \end{bmatrix} x_k.$$

Here C_{k+i} corresponds to the output matrix, while Φ_{k+i} represents the state transition matrix. Replacing Eq. 1b directly in the objective, leads to the following formulation:

$$\min_U J_2 = (E^* - S_m U)^T Q_1 (E^* - S_m U) + (U - U_r)^T Q_2 (U - U_r) \quad (2a)$$

$$\text{s.t. } U_l \leq U \leq U_u \quad (2b)$$

$$Y_{ld}^* \leq S_m U \leq Y_{ud}^*, \quad (2c)$$

where we have defined $E^* = Y_{sp} - Y^*$, $Y_{ld}^* = Y_l - Y^*$, and $Y_{ld}^* = Y_u - Y^*$.

For the unconstrained case, the analytical solution of the previous problem is:

$$U = (S_m^T Q_1 S_m + Q_2)^{-1}(S_m^T Q_1 E^* + Q_2 U_r) = K x_k + d_{sp} + d_r, \quad (3)$$

where

$$H = S_m^T Q_1 S_m + Q_2 \quad (3a)$$

$$K = -H^{-1} S_m^T Q_1 C^* \quad (3b)$$

$$d_{sp} = H^{-1} S_m^T Q_1 Y_{sp} \quad (3c)$$

$$d_r = H^{-1} S_m^T Q_2 U_r. \quad (3d)$$

In the above solution, K corresponds to a state feedback term, while d_{sp} and d_r can be seen as additional bias terms, denoting the fact that nonzero reference values are used for Y_{sp} and U_r. Clearly, only K contributes to the stability of the closed-loop system, since the remaining terms are fixed and bounded. The receding nature of MPC also requires that only the first move in computed profile be implemented at a time with the calculation repeated at the next sampling point, using any additional information available. This implies that the implemented manipulation at t_k is $u_k = [I_m \; 0 \; \cdots \; 0]U$, where U is given by the solution of either Eqs. 2 or Eq. 3.

Effect of Active Hard Constraints in the Closed-Loop Stability

In the presence of active hard constraints, the optimal input profile needs to be found as the solution of Eqs. 2. For a fixed (given) active set, we will denote the corresponding constraints as:

$$I^a U = U_b^a \quad (4a)$$

$$S_m^a U = Y_b^a - Y^{*a} = Y_b^a - C^{*a} x_k. \quad (4b)$$

The superscript a is introduced to denote active constraints. We represent the number of currently active input and output active constraints by n_u and n_y, respectively. In this case, the matrices $I^a \in \mathbf{R}^{n_u \times (n_i m)}$ and $S_m^a \in \mathbf{R}^{n_y \times (n_o p)}$ are obtained through selection of the rows of $I_{n_i m}$ and S_m that correspond just to the currently active constraints. Here U_b^a and Y_b^a represent the values of the active input and output bounds, respectively. These express either upper or lower bounds, or equality constraints which can also be specified for these variables. The above set of constraints can also be represented in a more compact form as:

$$A U = c_a, \quad (5)$$

where

$$A = \begin{bmatrix} I^a \\ S_m^a \end{bmatrix}, \quad c_a = \begin{bmatrix} U_b^a \\ Y_b^a - Y^{*a} \end{bmatrix}.$$

The effects of the presence of constraints are better illustrated with a range and null-space decomposition, performed on the matrix of constraints A. Since this matrix can be ill-conditioned, it is preferable to base the decomposition on a QR factorization of A^T with column pivoting (Golub and Van Loan, 1989). The advantage of this algorithm is that the determination of the numerical rank of the matrix to be factorized can be done simultaneously in a numerically robust form, taking into consideration the precision of the data available. Assuming that the rank of A is $n_r \leq n_i$ leads to:

$$A^T P = QR = [\mathcal{Q}_y \quad \mathcal{Q}_z] \begin{bmatrix} \mathcal{R} \\ 0 \end{bmatrix}, \tag{6}$$

where $Q \in \mathbf{R}^{n_i \times n_i}$ is an orthogonal matrix satisfying $Q^T Q = I_{n_i}$, and $\mathcal{R} \in \mathbf{R}^{n_r \times (n_u + n_y)}$ is an upper triangular matrix. $P \in \mathbf{R}^{(n_u+n_y) \times (n_u+n_y)}$ is a permutation matrix, obtained by interchange of the columns of the identity matrix of the same order. $\mathcal{Q}_y$ and $\mathcal{Q}_z$ correspond to a partition of the columns of Q with $\mathcal{Q}_y \in \mathbf{R}^{n_i \times n_r}$ and $\mathcal{Q}_z \in \mathbf{R}^{n_i \times (n_i - n_r)}$, respectively.

Premultiplying both members of Eq. 6 by Q^T gives $\mathcal{Q}_y^T A^T P = \mathcal{R}$ and $\mathcal{Q}_z^T A^T P = 0$. This implies that:

$$P^T A \mathcal{Q}_y = \mathcal{R}^T \quad \text{and} \quad A \mathcal{Q}_z = 0, \tag{7}$$

and therefore $\mathcal{Q}_z$ corresponds to a basis for the null-space of A. Hence, the full U space can be partitioned into range and null-space components as:

$$U = \mathcal{Q}_y U_y + \mathcal{Q}_z U_z, \tag{8}$$

where $U_y \in \mathbf{R}^{n_r}$ and $U_z \in \mathbf{R}^{n_i - n_r}$. The optimal solution for U can then be found separately in terms of these two components, which can be combined in the end according to Eq. 8 to give the optimal profile. It should be noted that since the U_y component is entirely determined by the current active constraints, the number of degrees of freedom in the input profile to be determined is just $n_i - n_r$.

We will start by calculating U_y, which can be obtained by replacing Eq. 8 in Eq. 5, giving $A \mathcal{Q}_y U_y = c_a$. This implies that $P^T A \mathcal{Q}_y U_y = P^T c_a$ or from Eq. 7:

$$\mathcal{R}^T U_y = P^T \begin{bmatrix} U_b^a \\ Y_b^a - Y^{*a} \end{bmatrix} = P^T c_a. \tag{9}$$

If $n_r = n_u + n_y$, that is, A is full row-rank, Eq. 9 constitutes a set of linear lower triangular equations which can be solved by forward substitution to obtain U_y. The solution can also be expressed analytically as

$$U_y = \mathcal{R}^{-T} P^T \begin{bmatrix} U_b^a \\ Y_b^a - Y^{*a} \end{bmatrix} = [B_y \ K_y] \begin{bmatrix} U_b^a \\ Y_b^a - Y^{*a} \end{bmatrix},$$

or

$$U_y = B_y U_b^a + K_y (Y_b^a - Y^{*a}), \tag{10}$$

where we have defined $[B_y \ K_y]$ as a compatible partition of the columns of $\mathcal{R}^{-T} P^T$. As with the unconstrained case, the contributions for U_y in Eq. 10 can be grouped into a bias term (given by $B_y U_b^a + K_y Y_b^a$), and a feedback term $-K_y Y^{*a}$. It should be noted that this last term will only appear in the optimal profile if there exist output constraints which are active.

However, if $n_r < n_u + n_y$, the matrix $\mathcal{R}^T$ will have a lower trapezoidal structure, and the linear system of Eq. 9 is over-determined, corresponding in general to a disjoint active set. In this case it is possible to create the partitions:

$$\mathcal{R}^T = \begin{bmatrix} \mathcal{R}_u^T \\ \mathcal{R}_l^T \end{bmatrix} \quad \text{and} \quad c_a = \begin{bmatrix} c_{au} \\ c_{al} \end{bmatrix},$$

where $\mathcal{R}_u^T \in \mathbf{R}^{n_r \times n_r}$ corresponds to the upper triangular part of $\mathcal{R}$, $\mathcal{R}_l^T \in \mathbf{R}^{(n_u+n_y-n_r) \times n_r}$ is the remaining rectangular matrix, and $c_{au} \in \mathbf{R}^{n_r}$, $c_{al} \in \mathbf{R}^{n_u+n_y-n_r}$ constitute an equivalent partition of c_a. This allows us to solve the lower triangular system $\mathcal{R}_u^T U_y = P^T c_{au}$, to find U_y. The computed solution can then be plugged back in the remaining equations, to check the feasibility of Eq. 9. If this set of equalities is compatible, then U_y can still be expressed in the form of Eq. 10, with $[B_y \ K_y]$ corresponding now to a column partition of $\mathcal{R}_u^{-T} P^T$. It should be noted that in this case the choice of active constraints which are retained in $\mathcal{R}_u$ doesn't affect the stability properties of the resultant controller (although it apparently can induce a different feedback structure), since all choices of constraints produce the same value of U_y.

On the other hand, the term U_z is determined by adjusting the remaining degrees of freedom in the profile to minimize the objective, if there are any left. From Eqs. 2, J_2 can be expressed as:

$$J_2(U) = c + 2a^T U + U^T H U, \tag{11}$$

where $a = -S_m^T Q_1 E^* - Q_2 U_r$ and H is given by Eq. 3a. Replacing Eq. 10 in Eq. 11 gives

$$J_2(U_z) = \text{constant} + 2a^T \mathcal{Q}_z U_z + 2U_y^T \mathcal{Q}_y^T H \mathcal{Q}_z U_z + U_z^T \mathcal{Q}_z^T H \mathcal{Q}_z U_z.$$

Solving $\nabla J_2(U_z) = 0$ leads to:

$$U_z = -(\mathcal{Q}_z^T H \mathcal{Q}_z)^{-1} \mathcal{Q}_z^T (a + H \mathcal{Q}_y U_y). \tag{12}$$

Equations 10 and 12 can finally be combined to give the analytical solution of the predictive control law. Replacing these in Eq. 8 gives:

$$U = K_h x_k + d_{h,sp} + d_{hr} + d_{hu} + d_{hy}, \tag{13}$$

where we have defined:

$$H_p = \mathcal{Q}_z^T (S_m^T Q_1 S_m + Q_2) \mathcal{Q}_z \tag{14a}$$

$$B = \mathcal{Q}_z H_p^{-1} \mathcal{Q}_z^T \tag{14b}$$

$$K_h = -(I_{n,m} - BH) \mathcal{Q}_y K_y C^{*a} - B S_m^T Q_1 C^* \tag{14c}$$

$$d_{h,sp} = B S_m^T Q_1 Y_{sp} \tag{14d}$$

$$d_{hr} = B Q_2 U_r \tag{14e}$$

$$d_{hu} = (I_{n,m} - BH) \mathcal{Q}_y B_y U_b^a \tag{14f}$$

$$d_{hy} = (I_{n,m} - BH)\mathcal{Q}_y K_y Y_b^a. \quad (14g)$$

Analogously to the unconstrained solution, the stability of the closed-loop law corresponding to a given active set depends only on K_h, since the other terms are fixed within a given active set, and are bounded for all sets of constraints. Comparing Eq. 13 with the corresponding unconstrained solution of Eq. 3, it is possible to observe that the general effect of the presence of constraints is to change the feedback structure of the system, replacing the unconstrained Hessian H by B, and introducing also additional bias terms d_{hu} and d_{hy}. In addition, the presence of active output constraints may yield extra feedback terms, as indicated by Eq. 14c. These feedback terms, which depend just on the current active set, can induce closed-loop instability in certain situations, as illustrated in the examples below. Also, as pointed out by Zafiriou (1990), we note that the stability properties of the predictive control law (Eq. 13) with hard constraints are identical for each of the upper and lower bounds in the process variables, since the feedback term in it is not influenced by the magnitudes of the bounds U_b and Y_b in Eqs. 4.

The global stability properties of the constrained system can now be considered. As indicated by Eq. 13, the overall controller is piecewise linear in x_k, with a structure which is only dependent on the current active set. This allows the use of the contraction mapping principle to show the following results.

Theorem 1 (*Economou, 1985*). Consider the augmented closed-loop system (Eq. 1) defined by the discrete LTV model $x_{k+1} = \Phi_k x_k + \Gamma_k u_k$, together with the state-feedback control law (Eq. 13), $u_{k+1} = \Psi_k x_k + \Upsilon_k u_k + d(y_{sp}, u_r)$. Define also an initial state $z_0 = [x_0^T \; u_0^T]^T$ and reference inputs $y_{sp,k+i} = y^*$, $u_{r,k+i-1} = u^*$, $i = 1, \ldots, p$. If:

$$\|F'(z)\| \equiv \left\| \frac{\partial X(T;\, x,\, u,\, y^*,\, u^*)}{\partial z} \right\| = \left\| \begin{matrix} \dfrac{\partial \chi}{\partial x} & \dfrac{\partial \chi}{\partial u} \\ \dfrac{\partial \psi}{\partial x} & \dfrac{\partial \psi}{\partial u} \end{matrix} \right\| \le \theta < 1,$$

$$\forall z \in \mathcal{B}(z_0, r),$$

where

$$r \ge r_0 \equiv \frac{\|X(T;\, z_0,\, y^*) - z_0\|}{(1-\theta)},$$

and

$$\mathcal{B}(z_0, r) \equiv \{z \in R^{n_s + n_i}: \; \|z - z_0\| \le r\},$$

then the system has a unique asymptotically stable equilibrium point $z_e = [x_e^T \; u_e^T]^T$ in $\mathcal{B}(z_0, r)$. Furthermore, $\mathcal{B}(z_0, r_0)$ is a region of attraction for z_e. (Here $\|\cdot\|$ represents any consistent norm definition.)

This theorem provides a sufficient condition for the stability of the closed-loop system, although its application with constrained systems is frequently limited in practice by the need of finding a consistent norm for different active sets. The next result is simpler to verify in the general case, even though it represents only a necessary condition for stability.

Theorem 2 (*Zafiriou, 1990*). Consider the augmented closed-loop system (Eq. 1) defined by the discrete LTI model $x_{k+1} = \Phi x_k + \Gamma u_k$, together with the state-feedback control law (Eq. 13), $u_{k+1} = \Psi x_k + \Upsilon u_k + d(y_{sp}, u_r)$, an initial state $z_0 = [x_0^T \; u_0^T]^T$ and reference inputs $y_{sp,k+i} = y^*$, $u_{r,k+i-1} = u^*$, $i = 1, \ldots, p$. This system can only be asymptotically stable in $\mathcal{B}(z_0, r)$ if:

$$\rho_{sr}[F'(z)] = \rho_{sr}\begin{pmatrix} \Phi & \Gamma \\ \Psi & \Upsilon \end{pmatrix} \le \theta < 1, \quad \forall z \in \mathcal{B}(z_0, r),$$

where $\rho_{sr}(\cdot)$ represents the spectral radius of a matrix, $\rho_{sr}(A) = \max_i |\lambda_i(A)|$.

As mentioned, this theorem provides just a minimal condition for global stability. Nevertheless, it is frequently used in practice, together with an implicit assumption about the rate of change of the system structure (the so called *slowly varying* systems (Vidyasagar, 1978)). For example, when considering the closed-loop response of a LTI model with a fixed setpoint, and sufficiently small disturbances such that no changes in the current active set occur, this last condition becomes also sufficient for stability, since a fixed active set induces a fixed control structure. Furthermore, since any induced norm constitutes an appropriate consistent norm, theorem 1 becomes also easier to apply in this situation. Both of these results will be used in order to assess the stability of the examples given below.

Special cases

We consider now some specific cases of Eq. 13, which introduce special characteristics in the closed-loop behavior, when some particular combinations of constraints are active.

Only Input Constraints Active. In this case, the only active constraints are of the form of Eq. 4a. Because of the special structure of the matrix of constraints, the range and null space decomposition is straightforward. For example, if:

$$I^a = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$(I^a)^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{Q}_y} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{Q}_z} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{R}/0}.$$

As can be observed, $\mathcal{Q}_y$ may be taken in this situation as $(I^a)^T$, and $\mathcal{Q}_z$ formed from the remaining columns of the correspondent identity matrix. From Eq. 8, this immediately implies that if there are no constraints active in the first interval of the horizon, the implemented control input u_k always comes from the $\mathcal{Q}_z$ (or null) space, and corresponds to the first components of Eq. 12. Also in this case, $\mathcal{R} = I_{n_a}$, and there's no need for column pivoting during the QR factorization of A. Furthermore, the projected Hessian H_p is simply formed by selection of the free columns and rows of H, according to Eq. 14a. The inverse of this matrix is later projected back into the full space during the computation of B, as indicated by Eq. 14b.

First Output Constraint Active, with Single-Input Systems. Here the bases for the range and null spaces of the constraint matrix need to be found in general through the standard decomposition algorithm described above. However since $\mathcal{Q}_z$ always corresponds to a basis for the null space of A, this implies that:

$$A\mathcal{Q}_z = \begin{bmatrix} a_{1,i_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,i_2} & 0 & \cdots & 0 \\ a_{1,i_2+1} & a_{2,i_2+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1,n_i-n_r} \\ z_{21} & z_{22} & \cdots & z_{2,n_i-n_r} \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = 0,$$

and therefore the first row of $\mathcal{Q}_z$ must be identically null, that is, $[z_{11}\ z_{12}\ \cdots\ z_{1,n_i-n_r}]=0$. From Eq. 8, this implies that the null space contribution to U is completely omitted in the implemented control u_k, and is determined by U_y alone. As a consequence, the normal feedback structure of the controller is suspended, and the only possible source of feedback in the closed-loop system will be any active output constraints, as indicated by Eq. 14c. If there are only active input constraints, then the system will essentially behave as open-loop. This situation corresponds therefore to one of the most severe losses of degrees of freedom that can be induced by the presence of active constraints. It may easily induce closed-loop instability in several situations, such as with open-loop unstable plants and systems with nonminimum phase characteristics. This behavior is observed in some of the examples presented below.

Maximum Rate Limits in the Inputs. Maximum bounds in the rate of variation of the manipulated variables corresponds also to a frequently used direct control objective in MPC; most of the existing implementations provide some facility for the treatment of these constraints. These limits are especially useful to prevent aggressive control moves caused by strong changes during the process operation. The corresponding constraints can be expressed in the form

$$|\Delta u_{k+i}| = |u_{k+i} - u_{k+i-1}| \le \Delta u_{\max} \quad i=0, \ldots, m-1,$$

with u_{k-1} given, and $\Delta u_{\max}$ representing the maximum allowed input move during a unique sampling interval. These equations can also be written in matrix form as:

$$-\Delta U_{\max} \le GU \le \Delta U_{\max},$$

where

$$G = \begin{bmatrix} 1 & 0\ 0\ 0 & \cdots & 0\ 0 \\ -1 & 1\ 0\ 0 & \cdots & 0\ 0 \\ 0 & -1\ 1\ 0 & \cdots & 0\ 0 \\ \vdots & \vdots\ \vdots\ \vdots & \ddots & \vdots\ \vdots \\ 0 & 0\ 0\ 0 & \cdots & -1\ 1 \end{bmatrix} \in R^{(n_i m)\times(n_i m)} \qquad (15a)$$

$$\Delta U_{\max,j} = \begin{cases} \pm u_{k-1} + \Delta u_{\max} & j=1 \\ \Delta u_{\max} & j=2, \ldots, m. \end{cases} \qquad (15b)$$

When active, these constraints become equalities of the form $G^a U = \Delta U^a_{\max}$, where G^a and $\Delta U^a_{\max}$ are formed from the active rows of G and $\Delta U_{\max}$. The effects of their presence on the closed-loop stability, for example, with a mixture of constraints of the other types can therefore be treated by the algorithm described previously. Applying a similar reasoning, it is possible to conclude that analogous to the effect of active absolute bounds on the inputs, the stability properties with rate constraints depend just on the given active set, and not on the bounds themselves. However, the tuning parameters used can certainly influence whether or not a given active set of this type can become active (that is, optimal). Also, we should notice that because of the special form of Eq. 15a, similar considerations to the constrained single-input system case can be made here. Having a rate constraint active during the first sampling interval will also cause the closed-loop system to behave essentially as open-loop during the same period of time. Hence, possible stability problems with unstable plants can be anticipated in this situation as well.

Examples

We illustrate now the application of the previous method in the elucidation of the constrained stability properties of MPC, with a number of simple process examples. These particular models were selected in order to provide a succinct view of the more common behaviors and stability effects that can result with active hard constraints. Additional examples can be found in Oliveira (1994). The results shown here were obtained through an implementation of the algorithm described above in the *Mathematica* language (Wolfram, 1991).

Example 1. We will start with the first-order plus time-delay SISO system given by the transfer function:

$$G(s) = \frac{e^{-0.06s}}{(s+1)}.$$

This corresponds to a frequently used class of models that describes chemical processes with relatively simple and slow dynamics. The equivalent discrete pulse transfer function obtained with a sampling time $T=0.1$ is:

$$HG(z) = \frac{0.03921\,(z+1.427)}{z(z-0.9048)}.$$

For this model, we consider only simple bounds in the input and output variables, of the form of Eqs. 4. The lengths of the predictive horizons used with it are $m=p=5$, with tuning parameters $Q_y=1$, $Q_u=0.01$. In Figure 1 we have represented the spectral radius of the resultant closed-loop system, for each possible active set that can occur with this type of bounds. In order to sweep all different combinations of constraints in a systematic form, the following procedure is used. We begin by appending the output horizon to the end of the input horizon, in order to form a unique extended horizon, as indicated in Figure 2. Then, starting at the left, we read the correspondent binary number obtained by assigning either the bit 0 or 1 to

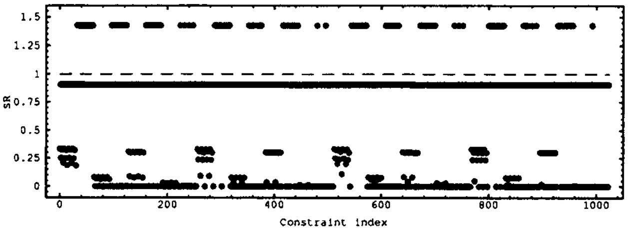

Figure 1. Closed-loop spectral radius (SR) of example 1 for all possible combinations of constraints.

the position of each variable in the extended horizon, depending on whether the correspondent constraint is inactive or active. For example, the total number of constraints in this case is $2^{10} = 1{,}024$. This means that the active set 0 will correspond to the unconstrained system; the active set 1 denotes the situation where only the first input in the first interval of the horizon is active; the active set number 32 represents the first output constrained only during the first interval; finally the last active set (1,024) represents the fact all input and output constraints are active simultaneously.

As can be observed, there exist certain active sets for this system which will make the resultant closed-loop system unstable. This is indicated by the existence of points above the unit spectral radius line in Figure 1. The problem occurs with the given tuning parameters, even though the spectral radius of the unconstrained system is about 0.33, well below the stability limit. A closer inspection of these situations reveals that the instability always occurs when the first output constraint in the horizon is active. This behavior was anticipated in the special cases considered before, since the discrete plant model has a zero outside the unit circle. Hence to avoid possible instability, this observation requires the use of a different approach for constraint handling, or even a system redesign such that these constraints will not become active during the normal plant operation. Furthermore, the example shows that all of the possible values of the closed-loop spectral radius are essentially grouped into various clusters, a fact that can be used to simplify significantly the elucidation of the stability properties of the constrained system. According to this observation, the most important information can usually be obtained through the investigation of only a much smaller subset of constraints than the total number of possible combinations. For instance, in this case, it would be sufficient to evaluate the stability properties of the system with all input constraints active throughout the horizon ($\rho_{sr} = 0.90$), all output constraints active throughout the horizon ($\rho_{sr} = 1.43$), and both sets simultaneously ($\rho_{sr} = 1.43$), together with the unconstrained information ($\rho_{sr} = 0.33$). This requires examining only

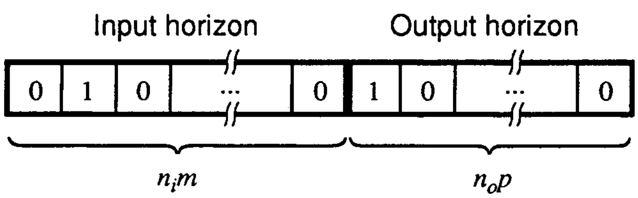

Figure 2. Building the constraint index.

4 active sets, corresponding to the cases where the same combination of inputs/outputs is assumed to be constrained throughout the horizon, instead of the full 1,024 cases. This guideline, although heuristic in nature, can potentially lead to tremendous reductions in the effort involved in a constrained stability analysis of a linear time-invariant (LTI) model. By eliminating the dependence on the length of the horizons, the number of cases which need to be considered can be reduced from $2^{n_o p + n_i m}$ to $2^{n_o + n_i}$, resulting usually in significant computational savings, given the typical size of the horizons used in practice. This heuristic applies also successfully to all examples considered so far, deserving therefore further attention in order to elucidate more rigorously its full range of validity.

Example 2. We examine now the stability properties of MPC applied to the linearized model of a Fluid Catalytic Cracking (FCC) unit. This model was obtained by linearization, followed by normalization, of the Lee and Kugelman (1973) model around a nominal stable operating point, as described in Oliveira and Biegler (1994). This example illustrates therefore the use of the previous methodology to study the local stability properties of a nonlinear model, around a fixed operating point. Using a sampling time $T = 0.05$ leads to the following discrete model:

$$x_{k+1} = \begin{bmatrix} 0.07362 & 0.1148 & 0.01044 & 0.4390 \\ 0.03940 & 0.1492 & 0.008940 & 0.8111 \\ 0.2794 & -0.7511 & -0.003253 & -6.127 \\ 0.04545 & 0.1559 & 0.009798 & 0.8384 \end{bmatrix} x_k + \begin{bmatrix} -0.2495 & 0.6030 \\ 0.01680 & 0.2505 \\ -5.785 & 7.2139 \\ 0.01157 & 0.1217 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_k,$$

where $x = [C_{sc}\ T_{rx}\ C_{rg}\ T_{rg}]^T$, and $u = [F_a\ F_c]^T$. Here C_{sc} denotes the coke content in the spent catalyst, T_{rx} the reactor bed temperature, C_{rg} the coke content in the regenerated catalyst, and T_{rg} is regenerator bed temperature. The manipulated variables are F_a (air flow rate) and F_c (catalyst circulation rate). The tuning parameters selected are $Q_y = \text{diag}\{1,\ 0.1\}$ and $Q_u = \text{diag}\{0.01, 0.01\}$. In order to limit the number of possible active sets, the lengths of the predictive horizons were fixed in this case as $m = p = 3$, with a total number of $2^{12} = 4{,}096$ distinct constraint sets. The spectral radius of the closed-loop system for each of these possible cases is plotted in Figure 3.

Several situations that may lead to closed-loop instability can also be observed in this figure. For instance, if the constraints in the first input u_1 become active throughout the predictive horizon, then the closed system has $\rho_{sr} = 1.12$. Similar behavior occurs if the second input saturates, leading in this case to $\rho_{sr} = 1.01$. The maximum value of spectral radius ($\rho_{sr} = 4.04$) is obtained however when both u_2 and y_1 are at their bounds. This behavior with the input constraints could not be easily anticipated just by examining the unconstrained characteristics of the plant, since the model is open-loop stable

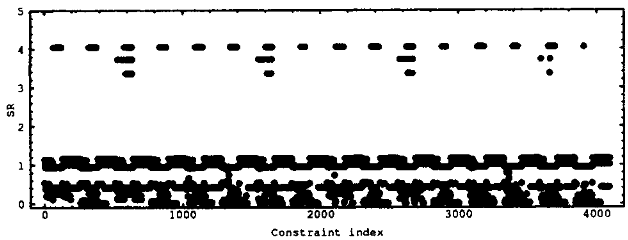

Figure 3. Closed-loop spectral radius (SR) of example 2 for all possible combinations of constraints.

and the unconstrained closed-loop system has $\rho_{sr}=0.53$. Looking now at the possible effects of the horizon lengths in the constrained stability shows that if m is kept equal to p, then the horizons need to be at least 7 intervals long to avoid problems associated with the saturation of u_1 (alone), and 16 intervals for case of u_2. These requirements are therefore much more restrictive than the needs for the simple stabilization of the unconstrained system. Nevertheless, the stability problem mentioned, that occurs when both u_2 and y_1 are constrained either during the first interval, or throughout the entire horizon, cannot be solved simply by increasing the length of the horizons used. If these constraints are active only during the first interval, it implies that the first two rows of Q_z are null, according to the special cases considered before; if they are active throughout the entire horizon, then the null space has zero dimension, and the solution is entirely determined by the range space component. As a result, the closed-loop spectral radius will be well above the stability limit in both cases. These active sets require therefore special attention in the design of the control system, so that they do not occur during normal operation of the process, or even the use of a different approach for constraint handling, like the soft constraint formalism described below.

Example 3. The last example consists of a laboratory system composed of two tanks and a connecting delay channel (Borrie, 1986). The continuous model for this system is described by the transfer function matrix:

$$G(s)=\begin{bmatrix}\dfrac{1.5e^{-0.2s}}{(s+3)} & \dfrac{1.5s}{(s+1)(s+3)}\\ \dfrac{3.0e^{-0.2s}}{(s+3)} & \dfrac{-3}{(s+3)}\end{bmatrix}.$$

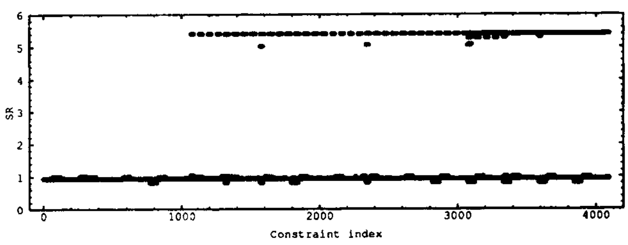

Figure 4. Closed-loop spectral radius (SR) of example 3 for all possible combinations of constraints.

Here the output variables represent the signals from the flow and pressure transducers, while the inputs represent the control signals to the upper and lower values, respectively. Using a sampling time of $T=0.07$ results in the following discrete model:

$$HG(z)=\begin{bmatrix}\dfrac{0.01478(z+5.409)}{(z-0.8106)z^3} & \dfrac{0.09136(z-1)}{(z-0.9324)(z-0.8106)}\\ \dfrac{0.02955(z+5.409)}{(z-0.8106)z^3} & \dfrac{-0.1894}{z-0.8106}\end{bmatrix}. \tag{16}$$

In order to use this model within the Newton framework, the discrete transfer matrix was first converted to a balanced state-space realization, using a standard technique based on a singular-value decomposition of the correspondent Hankel matrix (Chen, 1984). This resulted in a state-space model of order 6. We plotted in Figure 4 the spectral radius of the resultant closed-loop system, for predictive horizons of length 3, with $Q_y=\text{diag}\{1, 1\}$, $Q_u=\text{diag}\{0.1, 0.1\}$. A close look at these results reveal that there are also several active sets that can induce closed-loop instability with the above tuning parameters. Depending on the amount by which the stability limit is violated, these situations fall essentially into one of two different categories.

In the first group, we have active sets with ρ_{sr} slightly above the stability limit (typically less than 1.05). Examples of this group are for instance the constraint sets 000100001000_2 ($\rho_{sr}=1.004$), 000100001001_2 ($\rho_{sr}=1.017$), and 010000101001_2 ($\rho_{sr}=1.041$). Similar to the previous FCC example, the stability properties of these cases can be improved simply by increasing the lengths of the predictive horizons used. Furthermore, since they correspond to situations where a given variable is not constrained throughout the horizon, the likelihood of stability problems caused by their occurrence during normal operation is certainly low. In the second category, we have constraint sets with ρ_{sr} well above the unit spectral radius line (around 5.41). This behavior is first observed with the constraint set 010000101010_2, and also occurs whenever both of the outputs are saturated simultaneously. Here possible stability problems need to be considered more seriously, since they are not affected by the tuning parameters used, especially the lengths of the horizons. This is a consequence of the discrete model having a multivariable zero at $z=5.409$, revealed by constructing the canonical Smith-McMillan form of Eq. 16 (Kailath, 1980). Hence, special care is also needed to ensure that these hard constraints do not occur during normal operation, due to the incapability of MPC to handle them.

Strategies for Soft Constraint Handling

In order to remove the dependence of the closed-loop stability of MPC on certain active sets, several approaches have been proposed. Ricker et al. (1988) and Zafiriou (1991a) first suggested the use of a quadratic relaxation of the output constraints in the origin of these problems. Based on this philosophy, the last author derived a soft-constrained stability analysis for the QDMC framework, to assess the existence of potential constrained stability problems. As with the hard constrained case, this analysis suffers from the disadvantage of being combinatorial in the length of the horizon used. More recently, Rawlings and Muske (1993) proposed an alternative approach, based on the use of a fixed state-feedback law. This approach relies on the removal of the constraints that become infeasible in the beginning of the horizon. By keeping a finite input horizon, and extending the output horizon to infinity, it is shown that the resulting control law has guaranteed stability properties similar to the LQR (linear quadratic regulator) framework. Moreover, since the problem has only a finite number of degrees of freedom, it can still be solved on-line as a quadratic program, provided that an upper bound on the time for constraint feasibility is used. However, this bound is dependent on the controller itself, and is difficult to obtain without introducing considerable conservativeness in the measure itself. Additionally, the method does not provide a strict guarantee that the output constraints will eventually be enforced (especially in the presence of disturbances), due to the receding nature of MPC. In this group of approaches, one can also include the treatment of Mayne and Michalska (1990), which consists of the specification of a final equality constraint for the state vector at the end of the predictive horizon. This modification can be seen as a particular case of the Rawlings and Muske approach though, by noting that an infinite weight (equivalent to the specification of a final state constraint) is just a special case of the initial condition used for the recursive solution of the controller gain in the first method. As in the previous approach, guaranteed stability properties can be derived, provided that the constraint set remains always feasible.

In order to generalize the constraint treatment, we need to consider in more detail the consequences of a potential constraint violation in the process. Depending on their importance, process constraints can usually be classified as *hard* (if no violations are allowed at any time), or *soft* (where violations might be tolerated to satisfy their objectives). Examples in the first category include actuator limits, or safety constraints. In the second category we have, for example, output bounds corresponding to product specifications. While the original MPC formulation allows the specification of hard constraints, it might be preferable to consider in some situations a reformulation of (some of) the original constraints as soft, by introducing penalties associated with their noncompliance. Typical cases where this last treatment might be preferable include situations where:

- The original problem with hard constraints becomes infeasible. In this case it is not possible to satisfy exactly all constraints simultaneously. This problem is especially prominent in robust predictive formulations, where the constraints need to be enforced for an entire family of plants, parameterized by the uncertain parameters in the presence of disturbances (Campo and Morari, 1986, 1987). In this case, a heirarchy of objectives can be created by selecting the penalty weights according to the importance of satisfying each constraint individually.
- The original constrained system is unstable, for a significant range of the tuning parameters.

The examples presented in the previous section illustrate clearly the possible difficulties induced by the presence of active hard constraints (in some cases completely independent of the tuning parameters used), and motivate the need for a more general constraint handling methodology. The approach followed here is based on the use of penalty functions that allow for violations of some of these constraints by penalizing the correspondent violation in the objective. We start by considering the use of the more common quadratic (or exterior) penalty function, in the next section. This treatment is then extended to the l_1 and l_∞ penalty formulations in the section on exact penalty treatment. The perspective taken here is to concentrate first on the effect of the presence of output soft constraints alone, and then generalize in the end the results obtained to a mixture of both types of constraints. This view is consistent with the fact that output constraints tend to represent more easily desired control objectives, rather than hard limits in the process itself. Simultaneously, since the vast majority of the chemical processes is open-loop stable, their closed-loop stability properties are more sensitive to the presence of this type of constraints. Our main focus here will therefore be on the effects of output constraint softening on stability.

Constraint relaxation using the quadratic penalty

The presence of soft constraints is commonly associated with the use of the quadratic or l_2 penalty function (for example, Zafiriou, 1991b). Treating the output constraints as soft, the predictive control problem (Eqs. 2) can then be expressed as

$$\min_{U,\epsilon} J_{2a} = J_2 + \rho\epsilon^T\epsilon \tag{17a}$$

$$\text{s.t. } U_l \leq U \leq U_u \tag{17b}$$

$$Y_l - \epsilon \leq Y \leq Y_u + \epsilon \tag{17c}$$

$$\epsilon \geq 0, \tag{17d}$$

with J_2 given by Eq. 2a. Here $\rho \in \boldsymbol{R}^+$ is a scalar penalty parameter, while the vector $\epsilon \in \boldsymbol{R}^{n_o p}$ represents a measure of the original constraint violation. This formulation differs slightly from Zafiriou (1991b, 1992), where a full weight matrix W is used, together with the same scalar ϵ for all variables inside an arbitrary constraint window. Our analysis can also be extended to this formulation as well. However, the scalar formulation we use is closer to the one in the optimization literature, where it is known to have the following properties (Fletcher, 1987): when $\rho \to 0$, we obtain the unconstrained solution of Eqs. 2, when $\rho \to \infty$, the original (constrained) solution of Eqs. 2 is recovered; for finite values of ρ, the original active constraints are violated, which translates into the corresponding ϵ_i being positive.

The stability analysis developed here builds on the similarity of the present situation with the original unconstrained control law. As mentioned before, we will start by ignoring the presence of other types of constraints with the exception of the soft output constraints. In this case, the predictive problem (Eqs. 17) can be replaced by

$$\min_{U,\epsilon} J_{2a} = (E^* - S_m U)^T Q_1 (E^* - S_m U) + (U - U_r)^T Q_2 (U - U_r) + \rho\epsilon^T\epsilon \tag{18a}$$

$$\text{s.t. } Y_{ld}^* - \epsilon \leq S_m U \leq Y_{ud}^* + \epsilon \tag{18b}$$

$$\epsilon \geq 0. \tag{18c}$$

For a given active set, we can define $R = \text{diag}\{r_i\}$, $i = 1, \ldots, p$, with $r_i = \text{diag}\{r_{ij}\}$, $j = 1, \ldots, n_o$, such that:

$$r_{ij} = \begin{cases} 1 & j\text{th output constraint active at either} \\ & \text{the upper or lower bound, at } t_{k+i} \\ 0 & \text{otherwise.} \end{cases}$$

This definition allows us to eliminate the constraints (Eq. 18b and Eq. 18c) in the above problem, by substituting them directly in the objective function. Doing so yields the following unconstrained problem:

$$\min_U J_{2a} = J_2 + \rho (S_m U - Y^*_{bd})^T R (S_m U - Y^*_{bd}), \tag{19}$$

where $Y^*_{bd} = Y_b - Y^*$ is formed from the appropriate elements of Y_l or Y_u, depending on which constraints are active. This problem can be solved analytically, giving:

$$U_{\min} = (S_m^T(Q_1 + \rho R)S_m + Q_2)^{-1}(S_m^T Q_1 E^* + Q_2 U_r + \rho S_m^T R Y^*_{bd})$$
$$= K_a x_k + d_{ar} + d_{a,sp} + d_{ay}, \tag{20}$$

where

$$H_a = S_m^T(Q_1 + \rho R)S_m + Q_2$$

$$K_a = -H_a^{-1} S_m^T (Q_1 + \rho R) C^*$$

$$d_{ar} = H_a^{-1} Q_2 U_r$$

$$d_{a,sp} = H_a^{-1} S_m^T Q_1 Y_{sp}$$

$$d_{ay} = \rho H_a^{-1} S_m^T R Y_b.$$

A close look at the structure of the solution (Eq. 20) reveals the following result:

Theorem 3. The stability characteristics of the relaxed constrained problem (Eqs. 18) are equivalent to the stability of the unconstrained problem (Eq. 2a), with the tuning parameters $Q_1^* - Q_1 + \rho R$, $Q_2^* - Q_2$.

The proof for this theorem is given in the Appendix. This result equates the effects of the presence of soft constraints to an equivalent change in the tuning parameters of the unconstrained system. This corresponds to a more straightforward result than the stability criterion given by Zafiriou (1991a), because only unconstrained information is needed. However, since the term ρR depends on the current active set, different active sets have in general dissimilar effects on the closed-loop stability. Hence, as in the hard constraint case, a complete stability analysis requires a combinatorial study of the effects of different constraints, which can be difficult to perform for large horizons. Because the approach affects only output constraints, the total number of combinations that need to be considered now is just $2^{n_o p}$, though. As in the hard constraint case, the stability characteristics of the constrained control law identical for both bounds, since the feedback gain K_a does not depend on Y_b.

In addition, this approach requires knowing how the tuning parameters affect the closed-loop stability of the unconstrained system for a wide range of parameter values. This task can be readily accomplished for LTI systems, where a unique curve of the closed-loop spectral radius as a function of the penalty parameter ρ is needed. However, this information is more difficult to obtain with other types of models (such as LTV or nonlinear systems), where it becomes dependent on the initial conditions or on the operating region considered. The following examples illustrate the possible application of the analysis with simple LTI models.

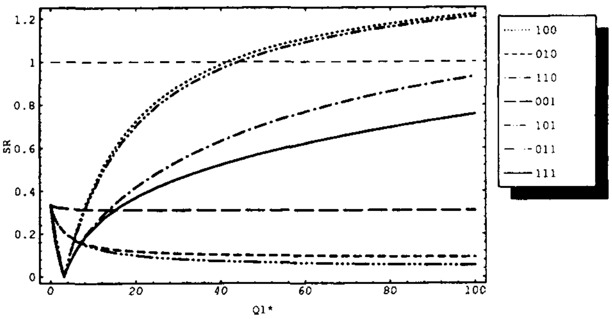

Figure 5. Effect of tuning parameter $Q_1^* = Q_1 + \rho R$ on the closed-loop spectral radius of example 1, for different active constraint sets.

Example 4. Consider the model defined in Example 1. Using predictive horizons of length $m = p = 3$, and nominal tuning parameters $Q_y = 1$, $Q_u = 0.01$, we plotted in Figure 5 the closed-loop spectral radius as a function of the penalty parameter ρ, for all possible active sets. In addition to the unconstrained information, we have $2^p - 1 = 7$ curves to check.

From Figure 5, we observe that closed-loop stability is guaranteed for $Q_1^* \le 41$ (approximately). This limit corresponds also to the case where the output constraint is just active in the first interval of the horizon. Therefore, this implies that if the output constraints for this example are relaxed as soft constraints, the penalty parameter used must obey $\rho < 40$ in order to avoid closed-loop instability with some of the present active sets.

Example 5 (Rawlings and Muske, 1993). We consider now the discrete, open-loop stable realization:

$$x_{k+1} = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k \tag{21a}$$

$$y_k = [-2/3 \;\; 1] x_k, \tag{21b}$$

obtained from the continuous system:

$$G(s) = \frac{42 - 11s}{s^2 + 4s + 42},$$

with a sampling time $T = 0.1$. We assume a initial condition $x_0 = [3 \;\; 3]^T$ and output constraints $|y_k| \le 0.5$, or equivalently for the state-space representation:

$$\begin{bmatrix} -2/3 & 1 \\ 2/3 & -1 \end{bmatrix} x_k \le \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

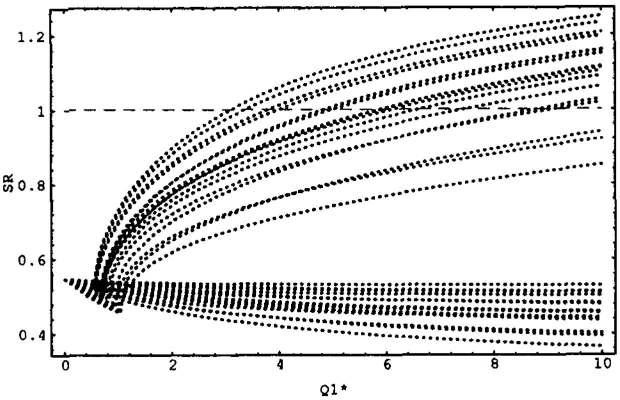

Figure 6. Effect of tuning parameter $Q_1^* = Q_1 + \rho R$ on closed-loop spectral radius of example 5, for different active constraint sets.

This example exhibits also stability problems when we try to enforce the output bounds given, as hard constraints. For example, since the initial condition violates the output constraint, from Eqs. 21 we need $u_0 \geq 1.75$ to satisfy this constraint at t_1. However, this condition causes also the state vector to increase in norm at t_1. This implies in turn that $u_1 > u_0$, in order to keep inside the feasible region. The input and state sequences will consequently increase in magnitude, in an unbounded form. Furthermore, including maximum bounds in the input, in the form $|u_k| \leq u_{max}$ will only make the constrained problem infeasible at some point in the future. Like most of the previous examples, this effect is independent of the tuning parameters used, since the control solution is always obtained from the range space of the constraints.

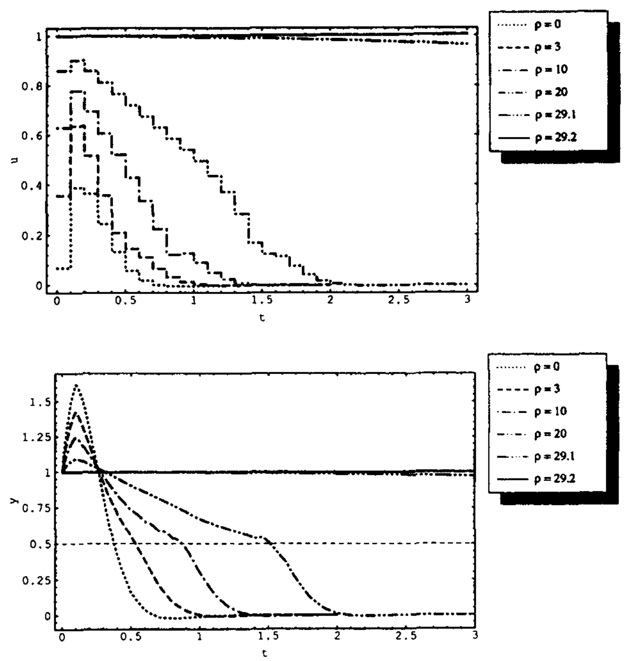

Figure 7. Input and output profiles for example 5.

Using a quadratic penalty relaxation of the output constraints, with horizons of length 5, leads to $2^5 - 1 = 31$ possible active sets to consider. For $Q_y = Q_u = 1$, we plotted in Figure 6 the effects of the penalty parameter ρ in the closed-loop spectral radius, for all possible constraint combinations. The stability limit is in this case $Q_1^* \leq 3.1$, corresponding again to the output constraint being active just during the first interval of the horizon.

Figure 7 illustrates the control profiles obtained with this constraint relaxation strategy. In all cases except with the highest value of ρ the system is stabilized, although this requires the output constraint to be violated during the first two intervals. To verify the conservativeness of the above stability limit, we tried to obtain numerically the minimum value of ρ that would make the closed-loop system unstable, from the given initial conditions. The value found in Figure 7 is $\rho_{critical} \in [29.1, 29.2]$, which is higher than the theoretical limit of ρ with all constraints active, respectively $\rho_{max} = 21.2$. The discrepancy between these values is due to the small value of x_0 used, since the spectral radius provides only a sufficient condition for stability. This difference can be decreased simply by increasing the norm of x_0, because the feedback and bias terms in the control law have roughly opposite effects in the magnitude of U, in the present case.

As mentioned previously, one of the main disadvantages of the quadratic penalty formulation is that, even for LTI systems, the determination of the corresponding stability limits becomes a nontrivial task for large predictive horizons. Instead of explicitly enumerating all possible combinations of constraints, the evaluation of these limits would be faster if the problem was formulated as an equivalent optimization problem. However, the resultant problem becomes difficult to solve, because the spectral radius is in general a nonlinear, nonconvex, and nondifferentiable function of the parameter ρ. Some methods under development for the constrained robustness analysis of LTI systems (Balakrishnan and Boyd, 1991; Young et al., 1992) show, however, some promise in this situation. Based on the use of a branch and bound algorithm, these methods are able to refine successively the estimates provided by approximate bounds, allowing the attention to be centered quickly in the regions of the parameter space that are of more importance. An alternative approach based on the reformulation of the problem in a robust framework has also been described by Zafiriou and Chiou (1992, 1993). In this case knowledge about the closed-loop structure for some specific sets of constraints has been used to rearrange the model in a form suitable for using the structured singular value as a parametric robustness tool. The main advantage of these developments is that a sufficient condition for stability (stronger than theorem 2) can be developed. A similar property for the exact penalty formulation (in this case independent of the value of the penalty parameter used) is derived in the next section.

The result obtained above can also be generalized to the case where a mixture of both hard and soft constraints are considered simultaneously. In order to do that, we define first the vectors Y_h and Y_s as the components of the augmented output vector Y for which hard and soft constraints are specified, respectively. In a set sense, we assume that $Y_h \cup Y_s = Y$, and $Y_h \cap Y_s = \emptyset$. Using a similar line of reasoning to the previous analysis, it is possible to derive the following result:

Theorem 4. Consider the constrained quadratic penalty formulation, with a mixture of hard and soft output constraints:

$$\min_{U,\epsilon} J_{2a,\text{mix}} = (E^* - S_m U)^T Q_1 (E^* - S_m U) + (U - U_r)^T Q_2 (U - U_r) + \rho \epsilon^T \epsilon \quad (22a)$$

$$\text{s.t. } U_l \le U \le U_u \quad (22b)$$

$$Y_{lh} \le Y_h \le Y_{uh} \quad (22c)$$

$$Y_{ls} - \epsilon \le Y_s \le Y_{us} + \epsilon \quad (22d)$$

$$\epsilon \ge 0. \quad (22e)$$

The stability properties of the mixed constrained problem (Eqs. 22) are equivalent to the stability of the correspondent hard constrained only problem (Eqs. 22a–c) and $\epsilon = 0$, with the tuning parameters $Q_1^* - Q_1 + \rho R$, $Q_2^* - Q_2$.

The proof for this theorem is given in the Appendix. It allows the stability properties of a predictive problem with mixed types of constraints to be related to the stability of the correspondent hard constrained only problem, for which the analysis presented in the third section can be applied. Using these results, it is therefore possible to perform a stability analysis of linear models in the presence of a large variety of constraints. The method provides also a systematic mechanism for the choice of appropriate values for the tuning parameters in order to avoid most of the problems described in the examples of the third section. Also, although the previous formulation was targeted to the use of soft output constraints, the same approach can be used to treat input constraints provided that the bounds that these represent can be relaxed. The analysis of this case using the methodology described previously is straightforward.

Exact penalty treatment

One of the properties of the quadratic penalty previously is that for finite values of ρ, the original output constraints may not be satisfied, which translates to nonzero values of ϵ. This means that a violation of the original constraints is unavoidable with this formulation. Replacing the penalty term by the l_1 (exact) penalty function eliminates the necessity of increasing the penalty parameter to infinity to recover the original constrained solution. A sufficient condition for this is to have $\rho > |\lambda|_\infty$, where λ is the vector of Lagrange multipliers of the constraints in the original problem (Fletcher, 1987). This formulation therefore allows better control of the errors resultant from constraint softening.

Using the l_1 penalty and starting by ignoring the presence of input constraints, allows us to express the predictive problem (Eqs. 1a–1d) as:

$$\min_U J'_{2b} = J_2 + r^T \max\{0,\ Y - Y_u\} + r^T \max\{0,\ -Y + Y_l\}, \quad (23)$$

with J_2 given by Eq. 2a. Here $r \in R^{n_o p} = [\rho \ \cdots \ \rho]^T$ is a vector of penalty parameters. The penalty terms in Eq. 23 can be rearranged, leading to the equivalent constrained formulation:

$$\min_{U,\epsilon} J_{2b} = J_2 + r^T \epsilon \quad (24a)$$

$$\text{s.t. } Y_l - \epsilon \le Y \le Y_u + \epsilon \quad (24b)$$

$$\epsilon \ge 0, \quad (24c)$$

with ϵ keeping its original definition in the fourth section. The solution of Eqs. 24 can also be obtained in analytical form, and expressed by:

$$U_{\min} = (S_m^T Q_1 S_m + Q_2)^{-1} (S_m^T Q_1 E^* + Q_2 U_r - S_m^T \lambda_1 / 2) = K x_k + d_{sp} + d_r + d_{by}, \quad (25)$$

where K, d_{sp}, and d_r are defined by Eqs. 3b–d, and

$$d_{by} = -(S_m^T Q_1 S_m + Q_2)^{-1} S_m^T \lambda_1 / 2.$$

The derivation of Eq. 25 is presented in the Appendix. Here λ_1 is a vector of Lagrange multipliers for the active constraints. The main difference with respect to the quadratic penalty treatment is that the multiplier λ_1 replaces now the penalty parameter r in the feedback law. The presence of each active set introduces consequently a different bias term d_{by} in the control law, dependent on the correspondent Lagrange multipliers. Since the multipliers are in turn dependent on the values of the states, Eq. 25 is actually a nonlinear controller. Nevertheless, it is also noted in the Appendix that $|\lambda_1|_\infty \le \rho$, which provides an upper bound on the magnitude of the term d_{by}. Thus, the states generated by Eq. 25 can be bounded by a linear controller with a gain matrix K given by Eq. 3b and a fixed bias term $d_\rho = -(S_m^T Q_1 S_m + Q_2)^{-1} S_m^T r/2$. We therefore have the following result:

Property 1. The control law (Eq. 25), for the exact penalty relaxation of the output constrained control problem with a finite ρ, has identical bounded stability properties to the corresponding unconstrained control law (Eq. 2a).

This result is derived in the Appendix, using the boundedness properties of d_{sp}, d_r, and d_{by}, and noting that the term Kx_k is identical to the unconstrained case. Furthermore, it is also shown that the bounds for the relaxed and unconstrained responses can be made arbitrarily close by decreasing the value of the penalty parameter used. Here an important difference with respect to the case of an unstable hard constraint behavior should be noted. Due to the form of Eq. 25, and assuming a stable unconstrained system, the closed-loop system can only become unstable if the multipliers λ_1 grow unbounded. However, this is not possible with a finite ρ, since we have always $\|\lambda_1\|_\infty \le \rho$. Therefore, the soft constrained solution will never be identical to the unstable hard constrained response with a finite value of ρ, although the difference between the two will decrease with increasing ρ. As a consequence, the approach becomes considerably simpler to apply, especially with time-varying and nonlinear systems, when compared with the previous quadratic constraint relaxation strategy. The following example illustrates this behavior.

Example 6. Consider again the linearized model of a FCC unit described in Example 2. The previous analysis indicated that stability problems can occur in the presence of hard constraints, if both u_2 and y_1 saturate simultaneously. This situation is illustrated in Figure 8, where we plotted the closed-

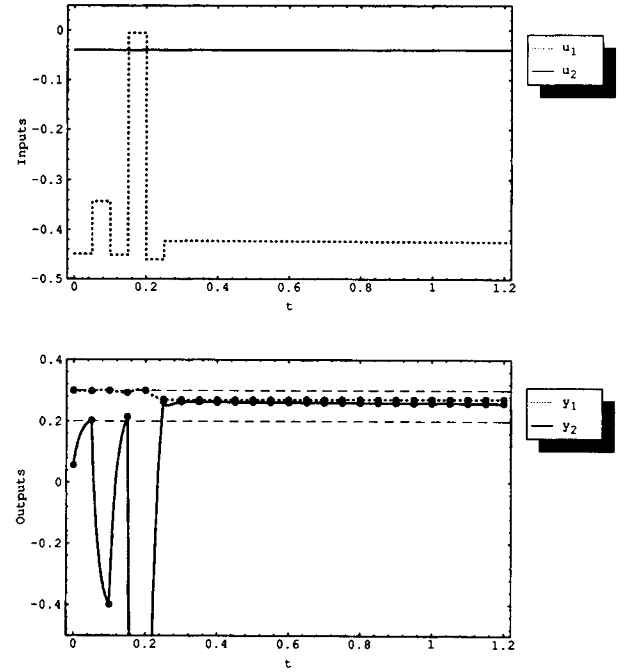

Figure 8. Input and output profiles for example 6 with hard constraints.

Dots indicate discrete sampling points considered.

loop response for $y_{sp}=[0.3\ 0.2]^T$, with $m=p=20$, using an initial condition $x_0=[0.2793\ 0.3000\ 0.0564\ 0.3200]^T$, and remaining parameters identical to the previous example. The hard constraints are in this case $u_2 \leq -0.0397$ and $y_1 \leq 0.3$. As can be observed, the controller starts by bringing y_2 closer to its setpoint, which induces the appearance of oscillation, since both of the hard constraints become active. This causes the outputs to move away from the setpoints after $t=0.25$ (because of the finite accuracy used in the computation of the solution). The outputs converge then very slowly to their respective reference values, indicating that this problem might occur again in the future.

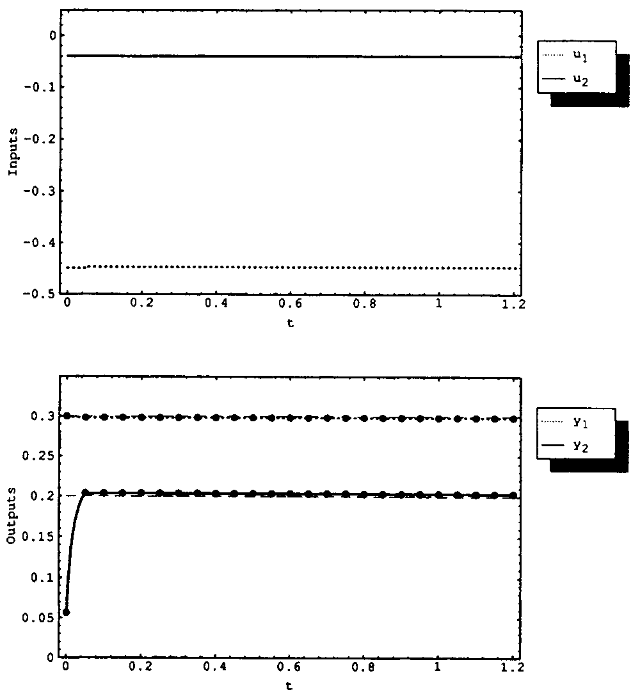

Figure 9. Input and output profiles for example 6 using l_1 penalty relaxation of output constraints.

The correspondent results using a l_1 penalty relaxation of the output constraint are shown in Figure 9. In contrast with the hard constraint situation, the profiles do not show any visible oscillation, and the desired setpoints are reached at the end of the first interval. This example illustrates therefore the ease of use of the l_1 penalty formulation. In contrast with the quadratic penalty formulation, no additional stability information is required now to choose the value of ρ (except knowing that the input only constrained system is also stabilized by the present tuning parameters).

A particular characteristic of this formulation is that in certain situations, as in the next example, large values of ρ can produce undesirable steady-state offsets in the closed-loop response, due to the receding nature of the MPC law. These offsets correspond also to a violation of the original constraints, but they can be eliminated simply by increasing the length of predictive horizon used. More precisely, this property can be stated as follows:

Property 2. The control law (Eq. 25), correspondent to the exact penalty relaxation of the output constrained control problem (Eq. 24), exhibits no steady-state offsets for a perfect model, and any finite value of the penalty parameter ρ, when the length of the output predictive horizon goes to infinity.

This property can be demonstrated by noting that with an infinite output horizon, the objective function (Eq. 24a) can only be made finite if the last input in the control profile is able to satisfy the limit of the setpoint trajectory y^*_{sp}, at some point in the output horizon. Therefore, as long as the setpoint is feasible and reachable, the optimal solution will have no permanent constraint violation, since the existence of at least one feasible point corresponding to a finite objective is guaranteed in this case. Hence, by choosing a sufficiently large output horizon, it becomes possible to use any value of ρ, in order to limit the error resultant from constraint relaxation. An additional way of eliminating these offsets is the use of integral action in the controller, for example, as considered by Oliveira and Biegler (1993). The following example illustrates this behavior with different horizon lengths.

Example 7. Consider again the system of Example 5. Using the same conditions as in the previous case, we plotted the control profiles correspondent to predictive horizons of length $m=p=5$ in Figure 10. The closed-loop system is stable for all values of the penalty parameter, although high values of ρ produce a steady-state offset, as mentioned. However, when the length of the predictive horizon is increased to 10 intervals (in Figure 11), this problem disappears, and all curves reach now the desired setpoint.

Similar to the quadratic penalty case, the stability properties with a mixture of both hard and soft constraints can be related back to the stability of the correspondent hard constrained only system. This is considered in the following theorem.

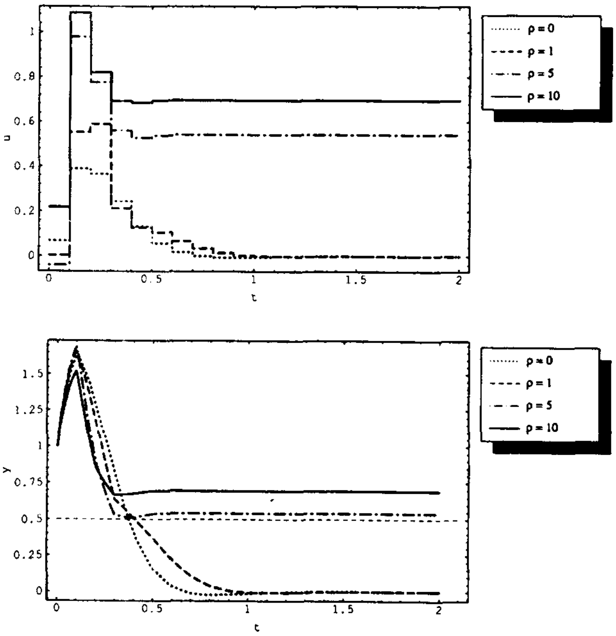

Figure 10. Input and output profiles for example 7.

Theorem 5. Consider the constrained l_1 penalty formulation, with a mixture of hard and soft output constraints:

$$\min_{U,\epsilon} J_{2b,\text{mix}} = (E^* - S_m U)^T Q_1 (E^* - S_m U) + (U - U_r)^T Q_2 (U - U_r) + r^T \epsilon \quad (26a)$$

$$\text{s.t. } U_l \leq U \leq U_u \quad (26b)$$

$$Y_{lh} \leq Y_h \leq Y_{uh} \quad (26c)$$

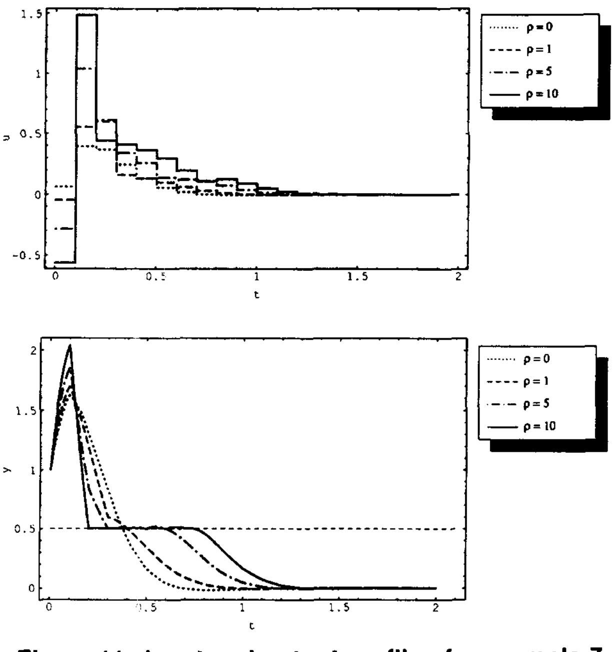

Figure 11. Input and output profiles for example 7.

$$Y_{ls} - \epsilon \leq Y_s \leq Y_{us} + \epsilon \quad (26d)$$

$$\epsilon \geq 0. \quad (26e)$$

The stability properties of the mixed constrained problem (Eqs. 26) are exactly identical to the stability of the correspondent hard constrained only problem (Eqs. 26a–c), with $\epsilon = 0$.

The proof for this theorem can also be found in the Appendix. The identical stability properties of the l_1 penalty to the hard constrained only case make therefore the exact penalty approach considerably easier to use than the quadratic penalty formulation, since this approach doesn't require the knowledge of how certain changes in the tuning parameters affect the stability of the correspondent constrained system. In addition, it should be noted that in contrast with theorems 3 and 4, property 1 and theorem 5 constitute actually sufficient conditions for (bounded) closed-loop stability of, respectively, the unconstrained or hard-constrained only system, since the linear feedback structure of the system is preserved now by the penalty formulation. Therefore, no additional assumptions about the rate of change of the system structure are required in this case for the application of theorem 2.

Constraint relaxation using other penalty formulations

The marked difference in the results derived in the previous sections for the two penalty formulations considered can be seen essentially as a consequence of the use of quadratic vs. linear penalty terms in the objective. The results obtained in this last case can therefore be expected to hold true for other penalty formulations which are linear in ϵ, such as the case of the l_∞ norm. In this case, the soft constraint formulation can be expressed as:

$$\min_{U,\epsilon} J_{2c} = J_2 + \rho\epsilon$$
$$\text{s.t. } Y_l - \ell\epsilon \leq Y \leq Y_u + \ell\epsilon$$
$$\epsilon \geq 0,$$

with $\ell = [1\ 1\ \cdots\ 1]^T$, and $\epsilon \in \mathbf{R}^+$ now. A similar analysis to the one developed in the previous section shows that properties 1 and 2, together with theorem 5 also apply in the present case. The main difference now is the bound for the Lagrange multiplier λ_1, which can be shown to satisfy $|\lambda_1|_1 \leq \rho$ instead. Otherwise, the control law is also given by Eq. 25. The performance of this formulation with the model considered in the Example 7 is illustrated in Figure 12, with horizons of length $m = p = 10$, and identical conditions to the previous case. The behavior shown is similar to the one obtained previously with the l_1 penalty (including the appearance of steady-state offsets with small horizons). However, since the infinity norm only weights the maximum constraint violation observed in the horizon, changes in ρ will induce now essentially the opposite effect in the characteristics of the closed-loop response. For instance making ρ larger will produce a lower peak, resulting in a slower output response, which stays outside the constraint bounds longer. This is a consequence of the inverse response nature of the system. We can also note that, contrary to the behavior of this system with the l_1 penalty, all profiles in Figure 12 for $\rho \neq 0$ have now a peak of smaller amplitude than the corresponding unconstrained response.

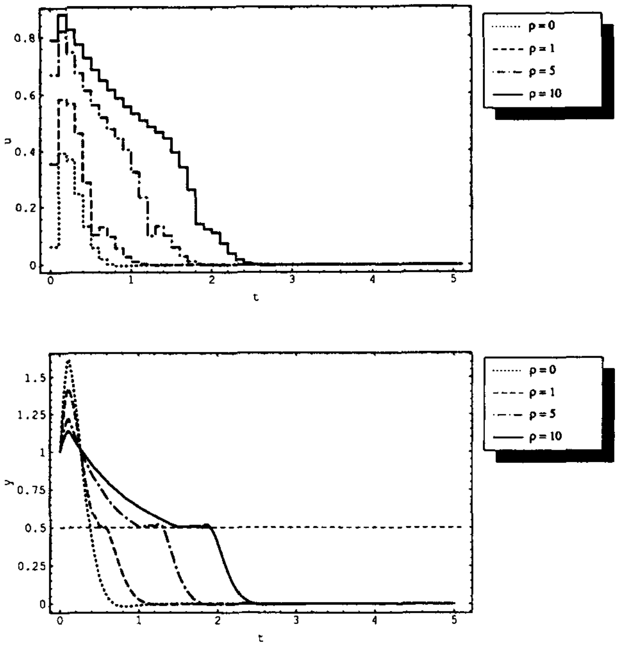

Figure 12. Input and output profiles for example 7 with l_∞ penalty.

Conclusions

This article presented a systematic analysis of the stability properties of MPC, in the presence of both hard and soft constraints. A perspective similar to treatment of Zafiriou (1990, 1991a,b) was used, which handles the nonlinearities introduced by the presence of active constraints through separate consideration of each different active set. Some important characteristics of the predictive control law were revealed here for the first time. These include the derivation of an explicit closed form expression for the optimal solution of the predictive problem in the presence of both hard and mixed types of constraints, as well as showing that active output constraints can introduce additional feedback terms in the constrained controller. As proposed above, the algorithm relies on a range and null space decomposition of the matrix of hard constraints, which is well suited for numerical (large scale) computation and can be implemented in a numerically robust form, for example, through a QR decomposition. The examples considered also show that in most cases the more important stability information relative to the presence of hard constraints can be derived by considering just a significantly smaller subset of constraints than the total number of possible combinations. According to this observation, the cases where the same combination of inputs/outputs is assumed to be constrained throughout the horizon give a good indication of whether hard stability problems can be expected. As seen, this allows a considerable reduction in the effort required for a constrained stability analysis of a linear model.

In addition to providing tools for systematic diagnosis of possible stability problems with hard constraints, this article presented some alternative constraint handling methodologies that enable these problems to be avoided. This is done through a relaxation of the problematic hard constraints, using a penalty formulation. This approach is most useful for output constraints, which represent frequently control objectives, rather than rigid limits in the process.

Starting with the case of a quadratic penalty, we showed that the stability properties with soft constraints can be related to the stability of the equivalent system with these constraints removed by a simple change in the tuning parameters used. This corresponds in general to a finite maximum value of the penalty parameter that can be tolerated for stability. Also, this approach still suffers from the disadvantage of being combinatorial in the length of the predictive horizon used. The exact (or l_1) penalty eliminates this last requirement leading to a constrained formulation that has the same bounded stability properties as in the absence of the soft constraints. Because of the nature of the problem, this last result extends also to other penalty formulations which are linear in the constraint violation term. This is especially true with the case of the l_∞ norm, for which a stronger bound for the Lagrange multipliers was shown to exist. This characteristic, together with the requirement of a finite value of the penalty parameter to match the solution of the original problem, simplifies considerably the use of soft constraints, especially for time-varying and nonlinear systems. We believe that the use of this exact penalty treatment has profound consequences on the design of constrained controllers, since it opens the possibility of using essentially all of the available tools for constrained control in this situation as well (for example, robustness, and so on). This observation is particularly pertinent to LTI models, for which a multitude of design methods is currently available.

Acknowledgments

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Appendix

Proof of Theorem 3

The optimality conditions for Eqs. 18 can be written as:

$$\mathcal{L} = J_2 + \rho\epsilon^T\epsilon + \lambda_u^T(S_mU - Y_{ud}^* - \epsilon) + \lambda_l^T(-S_mU + Y_{ld}^* - \epsilon) - \lambda_2^T\epsilon$$

$$\nabla_U\mathcal{L} = 2(S_m^TQ_1S_m + Q_2)U - 2S_m^TQ_1E^* - 2Q_2U_r + S_m^T(\lambda_u - \lambda_l) = 0 \quad \text{(A1a)}$$

$$\nabla_\epsilon\mathcal{L} = 2\rho\epsilon - \lambda_u - \lambda_l - \lambda_2 = 0 \quad \text{(A1b)}$$

$$\lambda_{2i}\epsilon_i = 0 \quad \text{(A1c)}$$

$$\lambda_{ui}(S_mU - Y_{ud}^* - \epsilon)_i = 0 \quad \text{(A1d)}$$

$$\lambda_{li}(-S_mU + Y_{ld}^* - \epsilon)_i = 0 \quad \text{(A1e)}$$

$$\lambda_{ui} \geq 0, \quad \lambda_{li} \geq 0, \quad \lambda_{2i} \geq 0, \quad \text{(A1f)}$$

with $i = 1, \ldots, n_op$. Assuming distinct upper and lower output bounds, the following combinations of values for the multipliers are possible.

(1) $\lambda_u = \lambda_l = \lambda_2 = 0, \quad \forall i = 1, \ldots, n_op$. This corresponds to the unconstrained case ($\epsilon = 0$).

(2) $\lambda_{ui} = \lambda_{li} = 0, \lambda_{2i} > 0$. From Eq. A1c, this implies that $\epsilon_i = 0$. However, from Eq. A1b, this implies that $\lambda_{ui} + \lambda_{li} < 0$, which contradicts the initial assumption. Hence, this combination of multipliers is not possible.

(3) $\lambda_{ui} > 0, \lambda_{li} = 0, \lambda_{2i} = 0$. From Eq. A1b, this implies that $\lambda_{ui} = 2\rho\epsilon_i$, and consequently $\epsilon_i > 0$. In this case we have from Eq. 19, $(S_mU)_i = Y_{ud,i}^* + \epsilon_i$, and Eq. 19 can be used in this case with $R_i \neq 0$, and $Y_{bd,i}^* = Y_{ud,i}^*$.

(4) $\lambda_{ui} = 0, \lambda_{li} > 0, \lambda_{2i} = 0$. From Eq. A1b, this implies that $\lambda_{li} = 2\rho\epsilon_i$, and consequently $\epsilon_i > 0$. In this case we have from Eq. 18b, $(S_mU)_i = Y_{ld,i}^* - \epsilon_i$, and Eq. 19 can be used also in this case with $R_i \neq 0$, and $Y_{bd,i}^* = Y_{ld,i}^*$.

(5) $\lambda_{ui} > 0, \lambda_{li} = 0, \lambda_{2i} > 0$. From Eq. A1c, $\epsilon_i = 0$, and Eq. A1b gives $\lambda_{ui} + \lambda_{2i} = 0$, which is inconsistent with the original assumption. This combination of multipliers is therefore not possible.

(6) $\lambda_{ui} = 0, \lambda_{li} > 0, \lambda_{2i} > 0$. Similarly to case 5, this leads to $\lambda_{li} + \lambda_{2i} = 0$, which is also inconsistent with the original assumption. Hence, this combination of multipliers is not possible.

Hence, Eq. 20 can be used with all consistent combinations of multipliers. This leads to:

$$\begin{aligned} U_{\min} &= H_a^{-1}(S_m^TQ_1E^* + Q_2U_r + \rho S_m^TRY_{bd}^*) \\ &= H_a^{-1}(S_m^TQ_1(Y_{sp} - Y^*) + Q_2U_r + \rho S_m^TR(Y_b - Y^*)) \\ &= H_a^{-1}(-S_m^T(Q_1 + \rho R)Y^*) + S_m^TQ_1Y_{sp} + Q_2U_r + \rho S_m^TRY_b), \end{aligned}$$

and therefore

$$U_{\min} = (S_m^T(Q_1 + \rho R)S_m + Q_2)^{-1}(-S_m^T(Q_1 + \rho R)C^*x_k + S_m^TQ_1Y_{sp} + Q_2U_r + \rho S_m^TRY_b). \quad \text{(A2)}$$

Comparing Eq. A2 with Eq. 3, we note that the feedback term becomes identical in both cases, if we replace Q_1 in Eq. 3 by $Q_1 + \rho R$. Therefore, the stability properties of the relaxed controller are identical to the equivalent unconstrained controller with the new tuning parameters.

Proof of theorem 4

Similar to the hard constraint stability analysis of the third section, we start by assuming that the present hard active set is known, and given by Eqs. 4. Also, as in the soft-constrained only case, we define $R = \text{diag}\{r_i\}$, $i = 1, \ldots, p$, with $r_i = \text{diag}\{r_{ij}\}$, $j = 1, \ldots, n_o$, such that:

$$r_{ij} = \begin{cases} 1 & j\text{th output soft constraint active at either bound, at } t_{k+i} \\ 0 & \text{otherwise.} \end{cases}$$

(Clearly, values of R different of zero will only occur now with the elements of Y_s.) This definition allows us to eliminate the soft constraints (Eq. 22d) in the mixed formulation, by substituting them directly in the objective function. Doing so yields the following hard constrained problem:

$$\min_U J_{2a,\text{mix}} = J_2 + \rho(S_mU - Y_{bd}^*)^TR(S_mU - Y_{bd}^*)$$
$$\text{s.t. } I^aU = U_b^a$$
$$S_m^aU = Y_b^a - Y^{*a}.$$

This problem can now be treated using the hard-constrained approach described in the third section. Since the hard constraints are identical in both cases, we will have the same bases for the range and null subspaces $\mathcal{Q}_y$ and Q_z in both situations, as well as the same range space solution U_y, given by Eq. 10. From Eq. 22a, the objective can be expressed as:

$$J_{2a,\,\text{mix}}(U) = \text{constant} + 2a_s^T U + U^T H_s U, \qquad \text{(A3)}$$

with $a_s = -S_m^T Q_1 E^* - Q_2 U_r - \rho S_m^T R Y_{bd}^*$, $H_s = S_m^T(Q_1 + \rho R) S_m + Q_2$. The first-order coefficient can also be expressed as:

$$a_s = -S_m^T Q_1 (Y_{sp} - Y) - Q_2 U_r - \rho S_m^T R (Y_b - Y^*)$$
$$= S_m^T (Q_1 + \rho R) Y^* - S_m^T Q_1 Y_{sp} - Q_2 U_r - \rho S_m^T R Y_b. \qquad \text{(A4)}$$

Replacing Eq. 8 in Eq. A3 gives:

$$J_{2a,\,\text{mix}}(U_z) = \text{constant} + 2a_s^T \mathcal{Q}_z U_z + 2U_y^T \mathcal{Q}_y^T H_s \mathcal{Q}_z U_z + U_z \mathcal{Q}_z^T H_s \mathcal{Q}_z U_z.$$

Solving $\nabla J_{2a,\,\text{mix}}(U_z) = 0$ leads to:

$$U_z = -(\mathcal{Q}_z^T H_s \mathcal{Q}_z)^{-1} \mathcal{Q}_z^T (a_s + H_s \mathcal{Q}_y U_y). \qquad \text{(A5)}$$

Equations 10 and A5 can finally be combined to give the analytical solution of the mixed constrained formulation. Replacing these equations in Eq. 8 gives:

$$U = K_s x_k + d_{h,sp} + d_{hr} + d_{hu} + d_{sy}, \qquad \text{(A6)}$$

where we have defined:

$$H_{ps} = \mathcal{Q}_z^T H_s \mathcal{Q}_z \qquad \text{(A7)}$$

$$B_s = \mathcal{Q}_z H_{ps}^{-1} \mathcal{Q}_z^T \qquad \text{(A8)}$$

$$K_s = -(I_{n_i m} - B_s H_s)\mathcal{Q}_y K_y C^{*a} - B_s S_m^T (Q_1 + \rho R) C^* \qquad \text{(A9)}$$

$$d_{sy} = (I_{n_i m} - B_s H_s)\mathcal{Q}_y K_y Y_b^a + \rho B_s S_m^T R Y_b, \qquad \text{(A10)}$$

with $d_{h,sp}$, d_{hr}, and d_{hu} defined by Eq. 14d–f. Comparing now Eq. A6 with Eq. 13, it is possible to observe that the feedback term becomes identical in both cases if we replace Q_1 in Eq. 13 by $Q_1 + \rho R$. Therefore, the stability properties of the mixed formulation are equivalent to the stability of the correspondent hard constraint only formulation, with the new tuning parameters.

Derivation of Eq. 25

We start with the optimality conditions of Eqs. 24:

$$\mathcal{L} = J_2 + r^T \epsilon + \lambda_u^T (Y^* + S_m U - Y_u - \epsilon) + \lambda_l^T (-Y^* - S_m U + Y_l - \epsilon) - \lambda_2^T \epsilon$$

$$\nabla_U \mathcal{L} = \nabla J_2 + S_m^T(\lambda_u - \lambda_l) = 2(S_m^T Q_1 S_m + Q_2) U - 2 S_m^T Q_1 E^* - 2 Q_2 U_r + S_m^T (\lambda_u - \lambda_l) = 0 \qquad \text{(A11a)}$$

$$\nabla_\epsilon \mathcal{L} = r - \lambda_u - \lambda_l - \lambda_2 = 0 \qquad \text{(A11b)}$$

$$\lambda_{2i} \epsilon_i = 0 \qquad \text{(A11c)}$$

$$\lambda_{ui}(Y - Y_u - \epsilon)_i = 0 \qquad \text{(A11d)}$$

$$\lambda_{li}(-Y + Y_l - \epsilon)_i = 0 \qquad \text{(A11e)}$$

$$\lambda_{ui} \geq 0, \quad \lambda_{li} \geq 0, \quad \lambda_{2i} \geq 0, \qquad \text{(A11f)}$$

with $i = 1, \ldots, n_o p$. From Eqs. A11b,f we conclude immediately that:

$$|\lambda_u|_\infty \leq \rho, \quad |\lambda_l|_\infty \leq \rho.$$

Assuming distinct upper and lower output bounds, together with the nondegenerancy of the problem (that is, there are no redundant constraints (either linearly dependent, or with the same solution as the unconstrained formulation)), the following combinations of values for the multipliers need to be considered:

(1) $\lambda_u = \lambda_l = \lambda_2 = 0$, $\forall i = 1, \ldots, n_o p$. This is inconsistent with Eq. A11b,f, and therefore this combination of multipliers is not possible.

(2) $\lambda_{ui} = \lambda_{li} = 0$, $\lambda_{2i} > 0$. From Eq. A11c, this implies that $\epsilon_i = 0$, and from Eq. A11b we have $\lambda_{2i} = \rho$. This corresponds therefore to the unconstrained case.

(3) $\lambda_{ui} > 0$, $\lambda_{li} = 0$, $\lambda_{2i} = 0$. From Eq. A11c, this implies that $\epsilon_i > 0$, and we will have some constraint violation. From Eq. A11b, we obtain $\lambda_{ui} = \rho$. In this case Eq. A11a becomes:

$$\nabla_U \mathcal{L} = \nabla J_2 + S_m^T \lambda_1 = 0, \qquad \text{(A12)}$$

with $\lambda_{1i} = \lambda_{ui}$.

(4) $\lambda_{ui} = 0$, $\lambda_{li} > 0$, $\lambda_{2i} = 0$. From Eq. A11c, this implies that $\epsilon_i > 0$, and we will have some constraint violation. From Eq. A11b, we obtain $\lambda_{li} = \rho$. In this case Eq. A12 applies with $\lambda_{1i} = -\lambda_{li}$.

(5) $\lambda_{ui} > 0$, $\lambda_{li} = 0$, $\lambda_{2i} > 0$. From Eq. A11c, $\epsilon_i = 0$. Also from Eq. A11d, $Y_i = Y_{ui}$, which corresponds to the exact solution of the original constrained problem. In this case Eq. A12 is also valid, with $\lambda_{1i} = \lambda_{ui} < \rho$.

(6) $\lambda_{ui} = 0$, $\lambda_{li} > 0$, $\lambda_{2i} > 0$. Similarly to case 5, this corresponds to the exact solution of the original constrained problem, now at the lower bound. Here Eq. A12 is also valid with $\lambda_{1i} = -\lambda_{li} < \rho$.

Hence, Eq. A12 is also valid for all consistent combinations of constraints. $\lambda_1 \in \mathbf{R}^{n_o p}$ can now be defined as:

$$\lambda_{1i} = \begin{cases} 0 & \text{if } \lambda_{ui} = \lambda_{li} = 0 \\ \lambda_{ui} & \text{if } \lambda_{ui} > 0 \\ -\lambda_{li} & \text{if } \lambda_{li} > 0 \end{cases}, \quad i = 1, \ldots, n_o p.$$

Solving Eq. A12 for U gives finally:

$$U_{\min} = (S_m^T Q_1 S_m + Q_2)^{-1} (S_m^T Q_1 E^* + Q_2 U_r - S_m^T \lambda_1 / 2).$$

Proof of property 1

The closed-loop response of a LTV system:

$$x_{s+1} = \Phi_s x_s + \Gamma_s u_s \tag{A13}$$

with the unconstrained controller (Eq. 3) can be written as:

$$x_{s+1} = \Phi_{cl,s} x_s + \Gamma_s d_s$$
$$x_{s+n} = \left(\prod_{j=1}^{n} \Phi_{cl,s+n-j}\right) x_s + \sum_{l=0}^{n-1} B^n_{s+l} d_{s+l}, \tag{A14}$$

where we have defined:

$$\Phi_{cl,s} = \Phi_s + \Gamma_s k_s$$
$$k_s = [I_m \ 0 \ \cdots \ 0] K_s$$
$$d_s = [I_m \ 0 \ \cdots \ 0](d_{sp,s} + d_{r,s})$$
$$B^n_{s+l} = \left(\prod_{j=1}^{n-l-1} \Phi_{cl,s+n-j}\right) \Gamma_{s+l}.$$

Without loss of generality, we assume in this derivation the use of $Y_{sp} = U_r = 0$, which corresponds to the equilibrium point $x^* = 0$, of the system (Eq. A13). If the unconstrained system is asymptotically stable, then $\lim_{n\to\infty} \Pi^n_{j=1} \Phi_{cl,s+n-j} = 0$, and a bound on the magnitude of x_{s+n} can be obtained by taking the norms of both sides of Eq. A14 giving, for sufficiently large n:

$$\|x_{s+n}\| \le \left\| \prod_{j=1}^{n} \Phi_{cl,s+n-j} \right\| \|x_s\| + \sum_{l=0}^{n-1} \|B^n_{s+l}\| \|d_{s+l}\|$$
$$\le C\eta^n + \xi \sum_{l=0}^{n-1} \eta^l \bar{d} \le C\eta^n + \frac{\xi \bar{d}}{1-\eta},$$

with $0 \le \eta < 1$, a finite $\xi > 0$, and $\bar{d} = \max_l \|d_{s+l}\|$, $l = 0, \ldots, n-1$. This bound remains therefore finite, as $n \to \infty$. Similarly, for the controller (Eq. 25) resulting from the exact penalty relaxation of the output constraints, we have:

$$x_{s+n} = \left(\prod_{j=1}^{n} \Phi_{cl,s+n-j}\right) x_s + \sum_{l=0}^{n-1} B^n_{s+l}(d_{s+l} + d_{b,s+l}),$$

with $d_{bs} = [I_m \ 0 \ \cdots \ 0] d_{by,s}$, which leads to the bound:

$$\|x_{s+n}\| \le C\eta^n + \bar{\xi} \sum_{l=0}^{n-1} \eta^l (\bar{d} + \rho)$$
$$\le C\eta^n + \bar{\xi} \frac{\bar{d} + \rho}{1-\eta},$$

also with finite $\bar{\xi}$, and where ρ is the penalty parameter defined in Eqs. 24. Therefore, the boundedness of the response of the unconstrained system implies also the boundedness of the state vector in the case of the exact penalty formulation.

Proof of Theorem 5

As previously, we start by assuming that the present hard active set is known, and given by Eqs. 4. Defining $Y_s = Y^*_s + S_{ms} U$ (that is, the outputs for which soft constraints are specified), the optimality conditions of Eqs. 26 can be written as:

$$\mathcal{L} = J_2 + r^T \epsilon + \lambda^T_{su}(Y^*_s + S_{ms}U - Y_{us} - \epsilon)$$
$$+ \lambda^T_{sl}(-Y^*_s - S_{ms}U + Y_{ls} - \epsilon)$$
$$+ \lambda^T_{hu}(I^a U - U^a_b) + \lambda^T_{hy}(S^a_m U - Y^a_b + Y^{*a}) - \lambda^T_2 \epsilon$$

$$\nabla_U \mathcal{L} = \nabla J_2 + (I^a)^T \lambda_{hu} + (S^a_m)^T \lambda_{hy} + S^T_{ms}(\lambda_{su} - \lambda_{sl}) = 0 \tag{A15a}$$

$$\nabla_\epsilon \mathcal{L} = r - \lambda_{su} - \lambda_{sl} - \lambda_2 = 0 \tag{A15b}$$

$$\lambda_{2i} \epsilon_i = 0 \tag{A15c}$$

$$\lambda_{su,i}(Y_s - Y_{us} - \epsilon)_i = 0 \tag{A15d}$$

$$\lambda_{sl,i}(-Y_s + Y_{ls} - \epsilon)_i = 0 \tag{A15e}$$

$$\lambda_{su,i} \ge 0, \quad \lambda_{sl,i} \ge 0, \quad \lambda_{2i} \ge 0, \tag{A15f}$$

with $i = 1, \ldots, n_{os}$, where n_{os} represents the total number of output variables for which soft constraints are specified. From Eqs. A15b,f it is also possible to conclude that:

$$|\lambda_{su}|_\infty \le \rho, \quad |\lambda_{sl}|_\infty \le \rho. \tag{A16}$$

Considering now the possible values of the multipliers λ_{su}, λ_{sl}, and λ_2 in a similar fashion to the derivation of Eq. 25 shows that Eq. A15a can also be expressed in the form:

$$\nabla_U \mathcal{L} = \nabla J_2 + (I^a)^T \lambda_{hu} + (S^a_m)^T \lambda_{hy} + S^T_{ms} \lambda_1 = 0, \tag{A17}$$

by defining in this case:

$$\lambda_{1i} = \begin{cases} 0 & \text{if } \lambda_{su,i} = \lambda_{sl,i} = 0 \\ \lambda_{su,i} & \text{if } \lambda_{su,i} > 0 \\ -\lambda_{sl,i} & \text{if } \lambda_{sl,i} > 0 \end{cases}, \quad i = 1, \ldots, n_{os}.$$

Comparing Eq. A17 with the optimality conditions of the hard-constraint only case, it is possible to observe that the only difference between these two cases is the presence of the additional bias term $S^T_{ms}\lambda_1$ in Eq. A17. Since this term is bounded in magnitude according to Eq. A16 and can also be treated by Property 1, the mixed constraint formulation possesses therefore identical stability properties to the correspondent hard constraint only formulation.

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